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Generalized α-Weakly Contractive Mappings on G-Metric Spaces

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ABSTRACT

In this paper we introduce the class of generalized α-weakly G-contractive mappings. We establish that such mappings have a unique common fixed point under certain weak conditions. The theorem obtained generalizes several recent results on generalized G-metric spaces relevant to Aydi *et al.* (2011).

Keywords: Common fixed point, generalized weakly G-contraction, generalized metric space.

1. INTRODUCTION

Albert and Guerre (1997) defined α-weakly contractive maps on Hilbegrt space and established a fixed point theorem for such map. Afterwards, Rhoades (2001), using the notion of weakly contractive maps, obtained a fixed point theorem in a complete metric space. Subsequently, many fixed points results of mapping satisfying certain contractive conditions have been studied by many authors (see Beg and Abbas (2006), Dutta and Choudhury (2008), Shatanawi (2010), Aydi *et al.* (2001) and Zheng and Song (2009)). Mustafa and Sims (2006) have introduced a new structure of generalized metric spaces, which are called G-metric spaces, as generalization of metric space (X,d). Recently, Aydi *et al.* (2011) have been established some common fixed point results for two self-mappings f and g on a generalized metric space X by assuming that f is a generalized weakly G-contraction mapping of types A and B with respect to g. In this paper, we generalized these results by given a weaker condition, taking a wide range of the constant α. Now we give preliminaries and basic definitions in G-metric spaces, which will be useful for the rest of the paper.

2. PRELIMINARIES

In this section, we introduce some basic notions and results that are used in the sequel. However, for more details, we refer to Mustafa and Sims (2006).

Definition 1. *Let X be a non-empty set. Suppose that $G: X \times X \times X \to R^+$ is a function satisfying the following conditions:*

$(G1)$ $G(x, y, z) = 0$ *find only if* $x = y = z = 0$ *(coincidence),*

$(G2)$ $0 < G(x, x, y)$ *for all* $x, y \in X$, *where* $x \neq y$,

$(G3)$ $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y \in X$, *with* $z \neq y$,

$(G4)$ $G(x, y, z) = G(p\{x, y, z\})$, *where* p *is a permutation of* x, y, z *(symmetry),*

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality).*

Then the function G is called the G-metric on X, and the pair (X, G) is called the G-metric space.

Definition 2. *Let (X, G) be a G-metric space. We say that (x_n) is*

(i) *A G-convergent sequence to $x \in X$ if, for any $\varepsilon > 0$, there is a positive integer k such that for all n, $m \geq k$, $G(x_n, x_m, x) < \varepsilon$;*

(ii) *A G-convergent sequence to $x \in X$ if, for any $\varepsilon > 0$, there is a positive integer k such that for all n, $m \geq k$, $G(x_n, x_m, x) < \varepsilon$;*

A G-metric space (X,G) is said to be complete if for every G-Cauchy sequence in X is G-convergent in X.

Proposition 3. *Let (X,G) be a G-metric space. The following are equivalent.*

(1) (x_n) *is* G*-convergent to* x*;*

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$;

(3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$;

(4) $G(x_n, x_m, x) \to 0$ *as* $n \to \infty$;

Proposition 4. *Let (X,G) be a G-metric space. The following are equivalent.*

(1) The sequence (x_n) *is* G*-Cauchy in* X;

(2) $G(x_n, x_m, x_m) \to 0$ (as $n,m \to \infty$).

Proposition 5. *Let (X,G) be a G-metric space. Then, for any $x, y, z, a \in X$ it follows that*

(i) *If* $G(x, y, z) = 0,$ *then* $x = y = z = 0;$

(ii) $G(x, y, z) \le G(x, x, y) + G(x, x, z);$

(iii) $G(x, x, y) \le 2G(y, x, x);$

(iv) $G(x, y, z) \le G(x, a, z) + G(a, y, z);$

(v) $G(x, y, z) \le G(x, a, a) + G(y, a, a) + G(z, a, a).$

Definition 3. *A G-metric space (X,G) is said to be symmetric if*

$$G(x, x, y) = G(x, y, y),$$

holds for arbitrary $x, y \in X$. if this is not the case, the space is called non-symmetric.

To every G-metric on the set X in a standard metric can be associated by

$$d_G(x,y) = G(x,x,y) + G(x,y,y).$$

If G is symmetric, then obviously $D_G(x,y) = 2G(x,x,y)$, but in the case of a non-symmetric G-metric, only

$$\frac{3}{2}G(x,y,y) \le d_G(x,y) \le 2G(x,y,y).$$

holds for all $x, y \in X$.

Shatanawi, Abbas and Aydi have introduced the concept of weakly G-contractive mappings as follows

Definition 4. *Let (X,G) be a G-metric space. A mapping $f: X \to X$ is said to be weakly G-contractive if for all $x,y,z \in X$, the following inequality holds*

$$\begin{aligned} G(fx,fy,fz) \le \frac{1}{3}&(G(x,fy,fy)+G(y,fz,fz)+G(z,fx,fx)) \\ &-\phi(G(x,fy,fy),G(y,fz,fz),G(z,fx,fx)), \end{aligned}$$

where $\phi:[0,\infty)^3 \to [0,\infty)$ is a continuous function with $\phi(t,s,u)=0$ if and only if $t=s=u=0$.

Khan *et al.* (1984) have introduced the concept of altering distance function as follows.

Definition 5. *The function $\psi:[0,\infty) \to [0,\infty)$ is called altering distance if the following properties are satisfied*

(1) *ψ is continuous and increasing*

(2) *$\psi(t)=0$ if and only if $t=0$.*

Aydi, Shatanawi and Vetro (2011) have introduced the following definition and theorem as follows.

Definition 6. *Let* (X,G) *be a* G*-metric space and* $f,g:X\to X$ *be two mappings. We say that* f *is a generalized weakly* G*-contractive mapping of type* A *with respect to* g *if for all* $x,y,z\in X$, *the following inequality holds*

$$\psi(G(fx,fy,fz))\le\psi(\frac{1}{3}(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))) \\ -\phi(G(gx,fy,fy),G(gy,fz,fz),G(gz,fx,fx)),$$

where ψ *is an altering distance function and* $\phi:[0,\infty)^3\to[0,\infty)$ *is a continuous function with* $\phi(t,s,u)=0$ *if and only if* $t=s=u=0$.

Theorem 1. *Let* (X,G) *be a* G*-metric space and* $f,g:X\to X$ *be two mappings such that* f *is a generalized weakly* G*-contractive mapping of type* A *with respect to* g. *Assume that* $f(X)\subseteq g(X),g(X)$ *is a* G*-complete subset of* (X,G) *and the pair* $\{f,g\}$ *is weakly compatible. Then* f *and* g *have a unique common fixed point.*

$$\psi(G(fx,fy,fz))\le\psi(\alpha(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))) \\ -\phi(G(gx,fy,fy),G(gy,fz,fz),G(gz,fx,fx)), \tag{1}$$

3. MAIN RESULTS

Definition 7. *Let* (X,G) *be a* G*-metric space and* $f,g:X\to X$ *be two mappings. We say that* f *is a generalized* α*-weakly* G*-contractive mapping of type* A *with respect to* g, *if for all* $x,y,z\in X$, *the following inequality holds*

where $\alpha\in[0,\frac{1}{3})$, ψ *is an altering distance function and* $\phi:[0,\infty)^3\to[0,\infty)$ *is a continuous function with* $\phi(t,s,u)=0$ *if and only if* $t=s=u=0$.

Now, let us introduce our main theorem.

Theorem 2. *Let* (X,G) *be a* G*-metric space and* $f,g:X\to X$ *be two mappings such that* f *is a generalized* α*-weakly* G*-contractive mapping* *of type A with respect to* g, *where* $\alpha \in [0, \frac{1}{3})$. *Assume that* $f(X) \subseteq g(X), g(X)$ *is a* G*-complete subset of* (X, G) *and the pair* $\{f, g\}$ *is weakly compatible. Then* f *and* g *have a unique common fixed point.*

Proof. Fix $x \in X$ and let $x_1 = x$. Then using $f(X) \subseteq g(X)$, we can construct a sequence $\{x_n\} : g(x_{n+1}) = f(x_n)$ for all $n \in N$. For simplicity, we can denote $g_{n+1} = gx_{n+1}$ and $f_n = fx_n$. We can assume that $g_{n+1} \neq g_n$ for all $n \in N$, otherwise f and g will have a common fixed point.

Using (1) and (G5) from Definition (1), we get

$$\begin{aligned} \psi(G(g_n, g_{n+1}, g_{n+1})) &= \psi(G(f_{n-1}, f_n, f_n)) \qquad (2) \\ &\leq \psi(\alpha(G(g_{n-1}, f_n, f_n) + G(g_n, f_n, f_n) + G(g_n, f_{n-1}, f_{n-1}))) \\ &\quad -\phi(G(g_{n-1}, f_n, f_n),\ G(g_n, f_n, f_n),\ G(g_n, f_{n-1}, f_{n-1})) \\ &\leq \psi(\alpha(G(g_{n-1}, g_{n+1}, g_{n+1}) + G(g_n, g_{n+1}, g_{n+1}) + G(g_n, g_n, g_n))) \\ &\leq \psi(\alpha(G(g_{n-1}, g_{n+1}, g_{n+1}) + G(g_n, g_{n+1}, g_{n+1}))), \end{aligned}$$

for all $n \in N$.

But ψ is an increasing function, thus from (2), we get

$$\begin{aligned} G(g_n, g_{n+1}, g_{n+1}) &\leq \alpha(G(g_{n-1}, g_{n+1}, g_{n+1}) + G(g_n, g_{n+1}, g_{n+1})) \qquad (3) \\ &\leq \alpha(G(g_{n-1}, g_n, g_n)) + 2\alpha G(g_n, g_{n+1}, g_{n+1}). \end{aligned}$$

For all $n \in N$, (3) gives the following

$$\begin{aligned} G(g_n, g_{n+1}, g_{n+1}) &\leq \frac{\alpha}{1-2\alpha} G(g_{n-1}, g_n, g_n) \qquad (4) \\ &\leq \left(\frac{\alpha}{1-2\alpha}\right)^2 G(g_{n-2}, g_{n-1}, g_{n-1}) \\ &\leq \ldots \leq \left(\frac{\alpha}{1-2\alpha}\right)^{n-1} G(g_1, g_2, g_2). \end{aligned}$$

Now, let us show that $\{g_n\}$ is a G-Cauchy sequence.

Letting $m > n$, then using (G5) in Definition (1) and (4), we get

$$\begin{aligned}0 < G(g_n, g_m, g_m) &\leq G(g_n, g_{n+1}, g_{n+1}) + G(g_{n+1}, g_{n+2}, g_{n+2}) \\ &\quad + G(g_{n+2}, g_{n+3}, g_{n+3}) + \ldots + G(g_{m-1}, g_m, g_m) \\ &\leq \left[\left(\frac{\alpha}{1-2\alpha}\right)^{n-1} + \left(\frac{\alpha}{1-2\alpha}\right)^{n} + \ldots + \left(\frac{\alpha}{1-2\alpha}\right)^{m-2}\right] G(g_1, g_2, g_2) \\ &\leq \left(\frac{\alpha}{1-2\alpha}\right)^{n-1}\left(1 + \frac{\alpha}{1-2\alpha} + \left(\frac{\alpha}{1-2\alpha}\right)^{2} + \ldots\ldots\right) G(g_1, g_2, g_2) \\ &\leq \left(\frac{1-2\alpha}{1-3\alpha}\right)\left(\frac{\alpha}{1-2\alpha}\right)^{n-1} G(g_1, g_2, g_2).\end{aligned} \tag{5}$$

Using the fact that $\alpha \in [0, \frac{1}{3})$ and letting $m \to \infty$, it follows that

$$\left(\frac{\alpha}{1-2\alpha}\right)^{n-1} \to 0 \text{ as } (n \to \infty).$$

Therefore,

$$G(g_n, g_m, g_m) \to 0 \text{ as } (n, m \to \infty).$$

Then by Proposition (4), we conclude that $\{g_n\} = \{gx_n\}$ is G-Cauchy in $g(X)$ which is G-complete subset of (X, G). Thus, there exist $z \in X$ such that gx_n is G-convergent to gz as $(n \to \infty)$. That is

$$G(g_n, g_n, gz) \to 0 \text{ as } (n \to \infty). \tag{6}$$

Also Proposition (3), gives us that

$$G(g_n, gz, gz) \to 0 \text{ as } (n \to \infty). \tag{7}$$

Now, we will show the following limits

$$\text{(a)} \quad G(g_n, g_n, fz) \to G(gz, gz, fz) \text{ as } (n \to \infty), \tag{8}$$

and

$$\text{(b)} \quad G(g_n, fz, fz) \to G(gz, gz, fz) \text{ as } (n \to \infty). \tag{9}$$

To show (a) use (iii) & (v) from Proposition (5), we get

$$\begin{aligned} G(g_n, g_n, fz) - G(gz, gz, fz) &\le 2G(g_n, gz, gz) \\ &\le 4G(g_n, g_n, gz). \end{aligned}$$

Also,

$$G(gz, gz, fz) - G(g_n, g_n, fz) \le 2G(g_n, g_n, gz).$$

Therefore,

$$\left| G(g_n, g_n, fz) - G(gz, gz, fz) \right| \le 4G(g_n, g_n, gz) \to 0 \text{ as } (n \to \infty).$$

To show (b). Using similar argument of (8), we will get the required result. Now, let us show that

$$fz = gz.$$

By (1), we have

$$\begin{aligned} \psi(G(g_{n+1}, g_{n+1}, fz)) &= \psi(G(f_n, f_n, fz)) \\ &\le \psi(\alpha(G(g_n, f_n, f_n) + G(g_n, fz, fz) + G(gz, f_n, f_n))) \\ &\quad -\phi(G(g_n, f_n, f_n), G(g_n, fz, fz), G(gz, f_n, f_n)) \qquad (10) \\ &= \psi(\alpha(G(g_n, g_{n+1}, g_{n+1}) + G(g_n, fz, fz) + G(gz, g_{n+1}, g_{n+1}))) \\ &\quad -\phi(G(g_n, g_{n+1}, g_{n+1}), G(g_n, fz, fz), G(gz, g_{n+1}, g_{n+1})). \end{aligned}$$

Letting $(n \to \infty)$. Using (8) and (9) and the continuities of ψ and ϕ, (10) becomes

$$\begin{aligned} \psi(G(gz, gz, fz)) &\le \psi(\alpha G(gz, fz, fz)) - \phi(0, G(gz, fz, fz), 0) \qquad (11) \\ &\le \psi(2\alpha G(gz, gz, fz)) - \phi(0, 2G(gz, gz, fz), 0) \\ &\le \psi(2\alpha G(gz, gz, fz)). \end{aligned}$$

It follows from Proposition (5) and the fact that ψ is an increasing function and $\alpha \in (0, \frac{1}{3}]$, we conclude from (11) that $G(gz, gz, fz) = 0$. Thus, $fz = gz$ and z is a coincidence point of f and g. But the pair $\{f, g\}$ is weakly compatible. Thus, letting $u = fz = gz$, we get $fu = gu$.

Now, we claim that $fu = gu = u$. To show this, (1) gives us the following inequality

$$
\begin{aligned}
\psi(G(gu, g_{n+1}, g_{n+1})) &= \psi(G(fu, f_n, f_n)) \\
&\leq \psi(\alpha(G(gu, f_n, f_n) + G(g_n, f_n, f_n) + G(g_n, fu, fu))) \\
&\quad -\phi(G(gu, f_n, f_n), G(g_n, f_n, f_n), G(g_n, f_{n-1}, f_{n-1})) \\
&= \psi(\alpha(G(gu, g_{n+1}, g_{n+1}) + G(g_n, g_{n+1}, g_{n+1}) + G(g_n, gu, gu))) \\
&\quad -\phi(G(gu, g_{n+1}, g_{n+1}), G(g_n, g_{n+1}, g_{n+1}), G(g_n, gu, gu)).
\end{aligned}
$$

Letting $(n \to \infty)$, we get

$$
\begin{aligned}
\psi(G(gu, gz, gz)) &\leq \psi(\alpha(G(gu, gz, gz) + G(gz, gu, gu))) \\
&\quad -\phi(G(gu, gz, gz), 0, G(gz, gu, gu)) \\
&\leq \psi(2\alpha G(gu, gz, gz)) - \phi(G(gu, gz, gz), 0, G(gz, gu, gu)) \\
&\leq \psi(G(gu, gz, gz)) - \phi(G(gu, gz, gz), 0, G(gz, gu, gu)),
\end{aligned}
$$

which is true if $\phi(G(gu, gz, gz), 0, G(gz, gu, gu)) = 0$, that is $g(u) = g(z) = u$. Therefore

$$u = g(u) = f(u).$$

Hence u is a common fixed point of f and g. To show uniqueness, let t be a another common fixed point of f and g (i.e. $t = g(t) = f(t)$). By 1, we have

$$\begin{aligned}\psi(G(u,t,t)) &= \psi(G(fu,ft,ft))\\ &\leq \psi(\alpha[G(gu,ft,ft)+G(gt,ft,ft)+G(gt,fu,fu)])\\ &\quad -\phi(G(gu,ft,ft),G(gt,ft,ft),G(gt,fu,fu))\\ &\leq \psi(\alpha[G(u,t,t)+G(t,u,u)])-\phi(G(u,t,t),0,G(t,u,u))\\ &\leq \psi(3\alpha G(u,t,t))-\phi(G(u,t,t),0,G(t,u,u))\\ &\leq \psi(G(u,t,t))-\phi(G(u,t,t),0,G(t,u,u)).\end{aligned}$$

Therefore, $\phi(G(u,t,t),0,G(t,u,u))=0$ and hence $G(u,t,t)=G(t,u,u)=0.$ Thus, $t=u.$

Corollary 1. *Let (X,G) be a G-metric space and $f,g:X\to X$ be such that*

$$G(fx,fy,fz)\leq \alpha(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx)),$$

where $\alpha\in[0,\frac{1}{3})$. Assume that $f(X)\subseteq g(X), g(X)$ is a complete subset of (X,G) and the pair $\{f,g\}$ is weakly compatible. Then f and *g have a unique common fixed point.*

Proof. We get the result by taking $\psi(t)=t$ and $\phi(t,s,u)=0,$ then apply Theorem 2.

Corollary 2. *Let (X,G) be a complete G-metric space and $f:X\to X$ be*

$$\begin{aligned}\psi(G(fx,fy,fz) &\leq \psi(\alpha(G(x,fy,fy)+G(y,fz,fz)+G(z,fx,fx)))\\ &\quad -\phi(G(x,fy,fy),G(y,fz,fz),G(z,fx,fx)),\end{aligned}$$

where $\alpha\in[0,\frac{1}{3})$, ψ is an altering distance function and $\phi:[0,\infty)^3\to[0,\infty)$ is a continuous function with $\phi(t,s,u)=0$ if and only if $t=s=u=0.$ Then f has a unique common fixed point.

Proof. We get the result by taking $g = Id$x, the identity on X, then apply Theorem 2.

Corollary 3. *Let (X,G) be a complete G-metric space and $f: X \to X$ be*

$$G(fx, fy, fz) \leq \alpha(G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx))$$
$$-\phi(G(x, fy, fy), G(y, fz, fz), G(z, fx, fx)),$$

where $\alpha \in [0, \frac{1}{3})$, ψ is an altering distance function and $\phi: [0,\infty)^3 \to [0,\infty)$ is a continuous function with $\phi(t,s,u) = 0$ if and only if $t = s = u = 0$. Then f has a unique common fixed point.

Proof. It follows by taking $\psi(t) = t$ in Corollary 2.

Definition 8. *Let (X,G) be a G-metric space and $f, g: X \to X$ be two mappings. We say that f is a generalized α-weakly G-contractive mapping of type B with respect to g, where $\alpha \in [0, \frac{1}{3})$ if for all $x, y, z \in X$, the following inequality holds*

$$\psi(G(fx, fy, fz)) \leq \psi(\alpha(G(gx, gx, fy) + G(y, fz, fz) + G(gy, gy, fz) + G(gz, gz, fx)))$$
$$-\phi(G(gx, gx, fy), G(gy, gy, fz), G(gz, gz, fx)),$$

where $\alpha \in [0, \frac{1}{3})$, ψ is an altering distance function and $\phi: [0,\infty)^3 \to [0,\infty)$ is a continuous function with $\phi(t,s,u) = 0$ if and only if $t = s = u = 0$.

Theorem 3. *Let (X,G) be a G-metric space and $f, g: X \to X$ be two mappings such that f is a generalized α-weakly G-contractive mapping of type B with respect to g, where $\alpha \in [0, \frac{1}{3})$. Assume that $f(X) \subseteq g(X), g(X)$ is a complete subset of (X,G) and the pair $\{f, g\}$ is weakly compatible. Then f and g have a unique common fixed point.*

Proof. It can be proven by using the same argument similar to Theorem 2.

Corollary 4. *Let* (X,G) *be a* G*-metric space and* $f,g:X\to X$ *be such that*

$$G(fx,fy,fz))\le(\alpha(G(gx,gx,fy)+G(gy,gy,fz)+G(gz,gz,fx)),$$

where $\alpha\in[0,\frac{1}{3})$. *Assume that* $f(X)\subseteq g(X), g(X)$ *is a complete subset of* (X,G) *and the pair* $\{f,g\}$ *weakly compatible. Then* f *and* g *have a unique common fixed point.*

Corollary 5. *Let* (X,G) *be a complete* G*-metric space and* $f:X\to X$ *be*

$$\psi(G(fx,fy,fz))\le\psi(\alpha(G(x,x,fy)+G(y,y,fz)+G(z,z,fx)))$$
$$-\phi(G(x,fy,fy),G(y,fz,fz),G(z,fx,fx)),$$

where $\alpha\in[0,\frac{1}{3})$, ψ *is an altering distance function and* $\phi:[0,\infty)^3\to[0,\infty)$ *is a continuous function with* $\phi(t,s,u)=0$ *if and only if* $t=s=u=0$. *Then* f *has a unique common fixed point.*

Corollary 6. *Let* (X,G) *be a complete* G*-metric space and* $f:X\to X$ *be*

$$G(fx,fy,fz)\le\alpha(G(x,x,fy)+G(y,y,fz)+G(z,z,fx))$$
$$-\phi(G(x,x,fy),G(y,y,fz),G(z,z,fx)),$$

where $\alpha\in[0,\frac{1}{3})$ and $\phi:[0,\infty)^3\to[0,\infty)$ *is a continuous function with* $\phi(t,s,u)=0$ *if and only if* $t=s=u=0$. *Then* f *has a unique common fixed point.*

4. EXAMPLES

In this section, we present some examples to demonstrate the validity of the hypotheses of Theorem 2.

Example 1. *Let* $X=[0,2]$, $G(x,y,z)=\max\{|x-y|,|y-z||z-x|\}$, *where*

$$\psi(t)=\frac{t}{2},\ \phi(t,s,u)=\frac{t+s+u}{k},\ k\geq\frac{2}{\alpha},$$

$fx=1$ *and* $gx=2-x$. *It can be asily shown that* f *is a generalised* α *-weakly* G*-contractive mapping of type* A *with respect to* g, *where and* $f(X)\subseteq g(X)$, $g(X)$ *is a complete subset of* (X,G) *and the pair* $\{f,g\}$ *is weakly compatible. Also,* $G(fx,fy,fz)=0$,

$$\psi(\alpha(G(gx,fy,fy))+G(gy,fz,fz)+G(gz,fx,fx)))=\frac{\alpha}{2}\left(|1-x|+|1-y|+|1-z|\right)$$

and

$$\phi(G(gx,fy,fy),G(gy,fz,fz),G(gz,fx,fx))=\frac{|1-x|+|1-y|+|1-z|}{k}.$$

Thus, all the conditions of Theorem 2 are satisfied in this example and so f *and* g *have a unique common fixed point, where* $x=1$ *is the unique common fixed point* f *and* g.

We construct here an example of non-symmetric G-metric that satisfies the hypotheses of Theorem 2.

Example 2. *(inspired by Choudhury and Maity (2011). Define a* G*-metric space* G *on a set* $X=\{a,b,c\}$ *by*

$$G(x,x,x)=0,\ x\in X,$$

$$G(a,a,b)=2,$$

$$G(a,a,c)=G(a,b,b)=G(b,b,c)=3,$$

$$G(a,b,c)=G(a,c,c)=G(b,c,c)=4,$$

with symmetry in all variables. Note that G *is non-symmetric since* $G(x,x,y)\neq G(x,y,y)$ *for* $x\neq y$. *Consider the mappings* $f,g:X\to X$ *given by* $f(a)=f(b)=f(c)=a$, $g(x)=x$, *for all* $x\in X$.

Take $\psi(t)=2t$ *and* $\phi(s,t,r)=\dfrac{s+t+r}{k}$, *where* $k\geq\dfrac{1}{2\alpha}$, *for* $\alpha\in[0,\dfrac{1}{3})$.

Denote $L=\psi(G(fx,fy,fz))$, $A=G(gx,fy,fy)$, $B=G(gy,fz,fz)$, $C=G(gz,fx,fx)$, $K=\psi(\alpha(A+B+C))-\phi(A,B,C)$.

(x,y,z)	A	B	C	L	K
(a,a,a)	0	0	0	0	0
(a,a,b)	0	0	2	0	$4\alpha-\dfrac{2}{k}$
(a,a,c)	0	0	3	0	$6\alpha-\dfrac{3}{k}$
(a,b,b)	0	2	2	0	$8\alpha-\dfrac{4}{k}$
(a,b,c)	0	2	3	0	$10\alpha-\dfrac{5}{k}$
(a,c,c)	0	3	3	0	$12\alpha-\dfrac{6}{k}$
(b,b,c)	0	2	3	0	$10\alpha-\dfrac{5}{k}$
(b,c,c)	0	3	3	0	$16\alpha-\dfrac{8}{k}$

Hence, all the conditions of Theorem 2 are fulfilled and it follows that f, g *have a unique common fixed point* $x=a$.

We construct her an example of a non-symmetric G-metric space that satisfy contractive condition in Definition 6 but not Definition 7.

Example 3. *Taking* $X=\{a,b,c\}$, *then defining* G*-metric on* X *as in Example 2 with* f,g *and* $\psi(t)=2t$ *but* $\phi(s,t,r)=\dfrac{s+t+r}{3}$.

Denote

$L=\psi(G(fx,fy,fz)),\ A=G(gx,fy,fy),\ B=G(gy,fz,fz),\ C=G(gz,fx,fx),$
$K=\psi(\alpha(A+B+C))-\phi(A,B,C).$

The following table shows that the contractive conditions fulfilled for some $\alpha, \frac{1}{6}\le\alpha<\frac{1}{3}$ *as in Definition 6 but not fulfilled for some* $\alpha, 0<\alpha<\frac{1}{6}$.

(x,y,z)	A	B	C	L	K
(a,a,a)	0	0	0	0	0
(a,a,b)	0	0	2	0	$4\alpha-\frac{2}{3}$
(a,a,c)	0	0	3	0	$6\alpha-1$
(a,b,b)	0	2	2	0	$8\alpha-\frac{4}{3}$
(a,b,c)	0	2	3	0	$10\alpha-\frac{5}{3}$
(a,c,c)	0	3	3	0	$12\alpha-2$
(b,b,c)	0	2	3	0	$10\alpha-\frac{5}{3}$
(b,c,c)	2	3	3	0	$16\alpha-\frac{8}{3}$

Hence, all the conditions of Theorem 2 are fulfilled and it follows that f, g *have a unique common fixed point* $x=a$.

5. APPLICATIONS

In this part, we will deduce that some common fixed point result for mappings satisfying the contractive of integral type in a complete G-metric space.

Let Γ be the set of all function $\lambda:[0,\infty)\to[0,\infty)$, satisfying the following conditions.

(i) λ is a Lebesgue integrable mapping on each compact subset of $[0,\infty)$,

(ii) For each $\varepsilon > 0$, $\int_0^{\varepsilon} \lambda(s)\ ds > 0$,

(iii) λ is subadditive on each $a, b \in [0,\infty)$.

We have the following result.

Theorem 4. *Let* (X,G) *be a complete* G*-metric space. Let* $f, g : X \to X$ *be two mappings such that* $f(X) \subseteq g(X), g(X)$ *is a* G*-complete subset of* (X,G) *and the pair* $\{f,g\}$ *is weakly compatible. If there exists* $\lambda \in \Gamma$ *such that for all* $x, y \in X$, *we have*

$$\int_0^{G(fx,fy,fz)} \lambda(s)\ ds \le \int_0^{\alpha(G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx))} \lambda(s)\ ds, \qquad (14)$$

for some $\alpha \in [0,\frac{1}{3})$. *Then* f *and* g *have a unique common fixed point.*

Proof. Take $\psi(t) = \int_0^{t} \lambda(s)\ ds$, $\phi(t,s,u) = 0$. It is an easy matter to see that the map $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function. Since inequality 14 satisfies the hypotheses of Theorem 2 it follows that f and g have a unique common fixed point.

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